QUANTIZATION AND PSEUDODIFFERENTIAL ANALYSIS

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The overused word "quantization" refers to a variety of activities originating from several domains of mathematics or mathematical physics: quantum mechanics, representation theory or, more generally, harmonic analysis, pseudodifferential analysis. It is our point of view that the most–embracing vantage point is the last one: since we fully realize that this may set us in a somewhat isolated position, we propose to argument all this with a variety of examples, putting emphasis, primarily, on the so–called composition formulas.

This survey is meant to present ideas, not techniques: this is one of the reasons why we have chosen, in all instances, to deal exclusively with the simplest case (claiming for generalizations, whether these have already been worked out or not), as characterized, for instance, in pseudodifferential analysis (*resp.* harmonic analysis, *resp.* modular form theory) by the sole consideration of the dimension one (*resp.* the group $SL(2,\mathbb{R})$, *resp.* the arithmetic group $SL(2,\mathbb{Z})$).

In 1926, H.Weyl first introduced his celebrated symbolic calculus, a linear correspondence from functions $f = f(y,\eta)$ of two variables to operators $\mathrm{Op}(f)$ acting on functions u of one variable. The formula

$$(\mathrm{Op}(f)\,u)(x) = h^{-1}\int_{\mathbb{R}^2} f\left(\frac{x+y}{2},\eta\right) e^{\frac{2i\pi}{h}(x-y)\eta}\,u(y)\,dy\,d\xi \tag{1}$$

had its source in the quantization problem from early quantum mechanics, which called for the definition of a correspondence from classical observables to quantum observables; at the same time, it paved the way for subsequent investigations in representation theory, as it made it possible, on the example of the Heisenberg representation, to isolate some of the features which, in the future, were to play in important role in general. Finally, it was also the first definition of a pseudodifferential calculus — in this domain, the function f is called the symbol of the operator $\mathrm{Op}(f)$ — and is mentioned as such in the foundational paper [4] of Kohn–Nirenberg: actually, when rediscovered for the sake of its applications to partial differential equations, pseudodifferential analysis, until at least the end of the seventies, relied on the use of the so–called standard calculus (more about it in a moment) rather than the Weyl calculus.

Applications of pseudodifferential analysis to partial differential equations make up an immense field: we shall have to ignore it completely, concentrating instead on some facts of structure of the calculus, mostly those generally ignored or not understood. Pseudodifferential analysis starts with a study of the way, under the map Op, properties of the symbol are transferred to properties of the operator: for instance [1], smooth symbols f with bounded derivatives of all orders exactly correspond to operators which are bounded in $L^2(\mathbb{R})$, and remain so after any number of commutations with operators taken from the pair Q, P of position and momentum operators. This gives a meaning to the sharp product $(f, g) \mapsto f\#g$ of symbols, characterized by the equation $\mathrm{Op}(f\#g) = \mathrm{Op}(f)\,\mathrm{Op}(g)$, an operation which will be the subject of a greater part of the present exposition.

The best-known composition formula is probably Moyal's expansion

$$(f_1\#f_2)(x,\xi) = \sum_{n\geqslant 0}\left(\frac{h}{4i\pi}\right)^n \sum_{j+k=n}\frac{(-1)^j}{j!\,k!}\left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial \xi}\right)^k f_1(x,\xi)\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial \xi}\right)^j f_2(x,\xi)\,, \qquad (2)$$

a series expansion of the closed "integral" formula

$$(f_1\#f_2)(X) = \{e^{i\pi h\,L}(f_1(X+Y)\,f_2(X+Z))\} \quad (Y=Z=0)\,, \qquad (3)$$

where L stands for the operator, on functions of $(Y;Z) = ((y,\eta);(z,\zeta))$, defined by

$$i\pi L = (4i\pi)^{-1}\Big\{-\frac{\partial^2}{\partial y\,\partial\zeta} + \frac{\partial^2}{\partial z\,\partial\eta}\Big\}. \qquad (4)$$

The latter formula is another way of writing

$$(f_1\#f_2)(X) = \frac{4}{h^2}\int_{\mathbb{R}^4} f_1(Y)\,f_2(Z)\,e^{-\frac{4i\pi}{h}[Y-X,Z-X]}\,dY\,dZ\,, \qquad (5)$$

a formula involving the symplectic form $[\ ,\ ]$ defined by

$$[(y,\eta),(z,\zeta)] = -y\zeta + z\eta\,. \qquad (6)$$

The equation (2) on one hand, the pair of equations (3) and (5) on the other hand, have not the same domain of validity. The first one is an ideal one when dealing with differential operators (the symbols of which are polynomials with respect to the second variable), the latter pair is more suitable when dealing with symbols with some degree of integrability.

The letter h stands for Planck's constant, the number which isolates the irreducible unitary representation π_h of Heisenberg's group under consideration within the corresponding series: we shall assume that the reader is familiar with all this, as well as with the concept of covariance: for instance, the Weyl calculus is covariant under the Heisenberg representation on one side, the action by translations of $\mathbb{R}^2$ (the quotient of Heisenberg's group by its center) on the *phase space* $\mathbb{R}^2$ on the other side. Of course, there is no genuinely greater generality in letting h take values different from 1, since this is tantamount to making a rescaling transformation in the phase space. Still, one is often interested, in partial differential equations, in problems involving small constants: this is the subject of the so-called *semi-classical analysis*, in which the connection from analysis to the geometry of the phase space (the *correspondence principle* in the terminology of the founding fathers of quantum mechanics) is more imediately apparent.

One of the points we wish to stress is that the role of a "small" parameter, assumed to play a role analogous to that of Planck's constant, in general quantization theory, has been overemphasized, and has led to serious misconceptions. In particular, despite some popular beliefs, expansions of the composition of two symbols as asymptotic power series in terms of one such parameter do not continue to hold in general. Next, there are in mathematical physics other small constants of interest (for instance c^{-1}), the role of which in quantum mechanics is just as important as that of Planck's constant. Finally, let us not forget that Taylor expansions do not exhaust mathematical analysis: in the spectral decomposition of self–adjoint operators, for instance, convergent integrals (with respect to the spectral parameter) and series have a more respectable role to play.

An alternative symbolic calculus of operators (still acting on functions of one variable) is the *standard*, or convolution–first, calculus, defined by the equation

$$(\mathrm{Op}_{\mathrm{std}}(f)\,u)(x) = \int_{-\infty}^{\infty} f(x,\xi)\, e^{2i\pi\, x\xi}\, \hat{u}(\xi)\, d\xi\,, \tag{7}$$

involving the Fourier transformation $u \mapsto \hat{u}$. We shall have to come back to it in a moment, as well as to the Wick–antiWick "calculus" (actually not really a calculus) defined as follows. Taking $h = 1$ for simplicity (as explained above, this is no loss of generality), we use Heisenberg's representation to introduce the family $(\phi_w)_{w\in\mathbb{C}}$ of *coherent states* on the real line, where $\phi_0(t) = 2^{1/4}\exp(-\pi t^2)$ is the normalized ground state of the standard harmonic oscillator $\mathrm{Op}\,(\pi(x^2+\xi^2))$ and, if $w = x+i\xi$, $\phi_w(t) = \phi_0(t-x)\cdot\exp\{2i\pi(t-\frac{x}{2})\xi\}$. The Wick symbol of an operator A on functions defined on the real line is the function $w \mapsto (\phi_w \mid A\,\phi_w)$ on the complex plane: an operator B is said to have an anti–Wick symbol g if the identity

$$Bu = \int_{\mathbb{C}} g(w)\,(\phi_w \mid u)\,\phi_w\, d\,\mathrm{Re}\, w\, d\,\mathrm{Im}\, w$$

holds. Now, very few operators do have an antiWick symbol, and to find an antiWick symbol from the Wick symbol of a given operator, one would have to solve a backwards heat equation: this is impossible in general, and disqualifies the Wick–antiWick calculus as a genuine symbolic calculus: *cf.* [7] for a discussion.

We now come to situations in which some homogeneous space G/H of the group $G = SL(2,\mathbb{R})$ is to play the role of the phase space. With standard notations, one can take $H = K = SO(2)$, or $H = MA$ (the subgroup of G of diagonal matrices), or $H = MN$. In connection with Kirillov's method of orbits, it is natural to conceive the first two homogeneous spaces as being the natural choices for symbolic calculi of operators acting on spaces of functions associated with the discrete (*resp.* the principal, or complementary) series of representations of G. So far as the nilpotent orbit G/MN is concerned, tradition has it that one should associate it with the so–called metaplectic representation of the twofold cover of G: in [11], it was found that, again, a series of representations (the *higher–level* metaplectic representations), rather than an isolated one, had better be considered.

In the case when the phase space is G/K, one considers the holomorphic discrete series of G or, more properly said, a prolongation of the projective discrete series of that group: it is then the family $\mathcal{D}_{\lambda+1}$, where the parameter λ, instead of being a positive integer, can be any real number > -1 (the case when $\lambda = 0$ corresponds to the Hardy space). When $\lambda > 0$, the Hilbert space $H_{\lambda+1}$ has a reproducing kernel: consequently, there is a natural family $(u_z)_{z\in G/K}$ of coherent states, which led Berezin [2, 3] to generalizing the Wick calculus alluded to above; the symbols that take the place of the antiWick (when existing) and Wick symbol of some operator are the contravariant and covariant symbols, in Berezin's terminology. The connecting map from the contravariant to the covariant symbol is an explicit function, in the spectral–theoretic sense, of the Laplace–Beltrami operator of the upper half–plane G/K and, as shown by the well–known asymptotics of the Gamma function on vertical lines, is just as bad as the operator $\exp \frac{\Delta}{4\pi}$ expressing the link between the anti–Wick and the Wick symbol in the flat case. Thus the Berezin calculus does not qualify as a genuine symbolic calculus of operators in any sense suitable for pseudodifferential analysis. A better choice is a generalization of the Weyl calculus, taking advantage, at each point $z \in G/K$, of the geodesic symmetry S_z around z and of the associated unitary map $\sigma_z = \mathcal{D}_{\lambda+1}(S_z)$: some phase factor, making this operator symmetric, has to be plugged in as well; replacing, in the definition of the anti–Wick (*resp.* Wick) symbol, the operator of orthogonal projection on the coherent state at z by the operator σ_z, one defines a new pair of symbols, the *active* and *passive* ones. The link from the active to the passive symbol of some operator is explicit as well [7], though this is no longer the case in the higher–rank situations, and behaves in a much better way, analytically, than the corresponding map from the Berezin calculus: indeed, both this operator and its inverse are nice "pseudodifferential" operators on the phase space, while the operator playing the role of the latter one is of infinite order in Berezin's case.

As explained above, no composition formula can exist in the Berezin calculus: however, there is an immediate integral formula expressing the *cováriant* symbol of the product of two operators with given *contravariant* symbols f_1 and f_2, of course not an associative operation: *formally* inverting the above correspondence and relying on Stirling's formula, Berezin also noticed the possibility of generalizing Moyal's formula (2) to a point, interpreting λ^{-1} as some kind of analogue of a Planck's constant. This is not going to be our point of view, primarily in view of the fact that, in the actual composition formulas one can build (several are possible), the composition of symbols always depends on λ in such a way that the point at infinity appears as an *essential singularity*, which makes expansions in terms of powers of λ^{-1} of limited value at the most: this does not contradict the fact that Berezin's expansion is formally correct up to error terms which are $O(\lambda^{-N})$ for every N.

Let us quote the following generalization [8] of (3), insisting on the fact that *all* symbols considered in the following formula are passive ones: $f_1 \# f_2$ is the symbol of the composition of two operators with symbols f_1 and f_2 in the calculus associated with $\mathcal{D}_{\lambda+1}$. Given a point z in the hyperbolic half–plane, one may set $q = \sinh r \cos\theta$, $p = \sinh r \sin\theta$, denoting as r the hyperbolic distance from i to z, and as θ the angle between the horizontal and the tangent at the point i to the hyperbolic line (a Euclidean circle) joining i to z: in this way one defines a chart Ψ_i from $\mathbb{R}^2$ to the half–plane; more generally, for every point $x+i\xi$, $\xi > 0$, one defines a chart $\Psi_{x+i\xi}$ from $\mathbb{R}^2$ to the half–plane by the equation $\Psi_{x+i\xi}(q,p) = \xi\, \Psi_i(q,p) - i\,x$. On the other hand, on functions of four variables $(q', p', q'', p'') = (P', P'')$, one defines the operator L, just as in (4), by the equation

$$i\pi\, L = (4i\pi)^{-1}\Big\{ -\frac{\partial^2}{\partial q'\,\partial p''} + \frac{\partial^2}{\partial p'\,\partial q''}\Big\}.$$

One can now generalize (3), starting with the consideration of the set of charts $(\Psi_{x+i\xi})$ and of the same operator L as before: only, instead of the exponential function $e^{i\pi hL}$, we must now consider the function $E_\lambda(-i\pi L)$ with

$$E_\lambda(z) = 4\pi \int_0^\infty J_\lambda(4\pi t)\, e^{-t^{-1}z}\, dt\,, \qquad z > 0\,, \tag{8}$$

in other words

$$
\begin{aligned}
E_\lambda(z) &= \sum \frac{(-1)^n}{n!} \frac{\Gamma(\frac{\lambda-n+1}{2})}{\Gamma(\frac{\lambda+n+1}{2})} (2\pi z)^n \\
&+ \frac{2\pi}{\sin \pi\lambda} \sum \frac{(-1)^n}{n!} \frac{(2\pi z)^{\lambda+2n+1}}{\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+2)} \,.
\end{aligned} \tag{9}
$$

Finally, the formula we have in mind (valid for nice symbols f_1, f_2: *cf.* [8]), to be compared to (3), reads

$$
(f_1 \# f_2)(x+i\xi) = \{\, E_\lambda(-i\pi L)\, ((f_1 \circ \Psi_{x+i\xi}) \otimes (f_1 \circ \Psi_{x+i\xi}))\,\} \quad (P' = P'' = 0)\,. \tag{10}
$$

The equation (9) clearly shows that, as a function of λ, the right–hand side of (10) has an essential singularity at infinity, while simultaneously explaining why this singularity cannot be observed in one neglects terms formally of the order $O(\lambda^{-\infty})$: note that the series in (9) is a convergent, not an asymptotic one. The paper just quoted also shows that, up to a point that increases to infinity with λ, the correspondence based on the use of, say, the passive symbol, makes it possible to trace properties of continuity of the operators under consideration through corresponding properties of their symbols: again, this is in striking contrast with the Berezin calculus.

With a view towards building a symbolic calculus of operators acting on functions in the Hilbert space of a representation taken from the principal series $\pi_{i\lambda}$, it is natural to use the one–sheeted hyperboloid G/MA as a phase space. The same defining formula was reached by Molchanov in [6] and ourselves in [9], though by two different means: in the first case, as a generalization of Berezin's quantization, in the second one as a generalization of the standard calculus (7), only replacing the Fourier transformation by the intertwining operator from $\pi_{i\lambda}$ to $\pi_{-i\lambda}$. The integral composition formula is immediate in this case, just as the one from the Berezin calculus: but it does not suffer the same defects, as the link between the two species of symbols to be considered this time is an invertible operator, even a unitary one. There is a subtler composition formula, valid for a part of the calculus, that concerned with an algebra generated by Hilbert–Schmidt operators which are inverses of differential operators of the kind $d\pi_{i\lambda}(X)$, $X \in \mathfrak{g}_{\mathbb{C}}$. The corresponding space of symbols is exactly the Hilbert sum of all eigenspaces of the basic invariant operator $\Box$ on $L^2(G/MA)$ (the *pseudo–* Laplacian) sitting discretely in the spectral decomposition of this operator. The formula can be found in [9], and has two interesting features: first, it gives a convergent series expansion, a situation one does not often come across, even in the flat case; next, the various terms are, up to explicit constants, related to the celebrated Rankin–Cohen brackets, once transferred as functions on the upper half–plane. It is important to note that, again, the coefficients are given as products of Gamma factors, certainly not as powers of λ^{-1}.

Going back to the case of G/K, one may raise the question whether one could make a kind of limiting calculus, as $\lambda \to \infty$, available: note that this has nothing to do with the deformation point of view; on the contrary, we wish to get rid of λ entirely. The representation $\mathcal{D}_{\lambda+1}$ has a limit or, more properly said, a contraction, which is a representation not of the group $G = SL(2,\mathbb{R})$, but of the Poincaré group of smallest dimension 3. This *Fuchs* calculus led to the construction of symbolic calculi compatible with the principles of special relativity, to wit the so–called *Klein–Gordon* and *Dirac* calculi, a very brief exposition of which may be found in [10]. The Klein–Gordon calculus can be made to depend on two constants (h and c), and its non–relativistic limit (*i.e.,* its limit as $c \to \infty$) is just the Weyl calculus.

Finally, we wish to come back to the one–dimensional Weyl calculus, in order to show that, even in that case, the already well–known composition formulas are far from being the end of the story. The Moyal formula, for instance, relies on the use of Taylor expansions, or of homogeneous polynomial symbols (possibly polynomial only with respect to the second variable). Now, if you start from the consideration, for instance, of two symbols such as fractional powers $\ell^{\frac{-1-\nu_1}{2}}$ and $\ell^{\frac{-1-\nu_2}{2}}$ of the symbol $\ell(x,\xi) = x^2+\xi^2$, it is a very bad idea to use Moyal's formula to compute their composition f since, while repeated differentiation improves the properties of the corresponding power of ℓ near infinity, it deteriorates them near zero. The correct formula must, of necessity, use the *continuous* decomposition of symbols into their homogeneous parts, and reads

$$f = \int_{-\infty}^{\infty} f_\lambda \, d\lambda$$

with

$$\begin{aligned} f_\lambda &= \frac{1}{4}(2\pi)^{\frac{\nu_1+\nu_2-i\lambda-1}{2}} \, \ell^{\frac{-1-i\lambda}{2}} \times \\ &\times \frac{\Gamma(\frac{1+\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{1+\nu_1-\nu_2+i\lambda}{4})\,\Gamma(\frac{1-\nu_1+\nu_2+i\lambda}{4})\,\Gamma(\frac{1-\nu_1-\nu_2-i\lambda}{4})}{\Gamma(\frac{1+\nu_1}{2})\,\Gamma(\frac{1+\nu_2}{2})\,\Gamma(\frac{1-i\lambda}{2})} . \end{aligned} \tag{11}$$

This was shown in [11, section 17]. It is a particular case of a general composition formula, in the Weyl calculus, calling for the decomposition of the #–product of any two homogeneous symbols as *integral* superpositions of homogeneous symbols. As explained in (*loc.cit.,* [section 19]), this is the correct point of view when putting the emphasis, in the Weyl calculus, on the covariance under the metaplectic representation rather than on that under the Heisenberg representation.

This new point of view makes it possible to develop a calculus of operators with extremely singular symbols, the consideration of which would be totally impossible with a more classical point of view. In particular, one can consider symbols invariant under the linear action on $\mathbb{R}^2$ of the group $SL(2,\mathbb{Z})$: of necessity, these *automorphic* symbols have to be distributions, rather than functions: it was shown in *loc.cit.* that they make up a simultaneous realization of all spaces of non–holomorphic modular forms (to be absolutely correct, of the more precise Lax–Phillips space [5]). The trick consists in associating with an even distribution $\mathfrak{S}$ on $\mathbb{R}^2$ the pair $(h_0,\, h_1)$ of functions on the upper half–plane defined as follows: starting from the pair $(\psi^0,\, \psi^1)$ of the first two normalized eigenstates of the standard harmonic oscillator on the line, associate with any point $z = g.i$ ($g \in SL(2,\mathbb{R})$) of the upper half–plane the pair of functions $\psi_z^j = \mathrm{Met}(g)\,\psi^j$, where $\mathrm{Met}(g)$ is a metaplectic transformation above g (the indeterminacy by a unitary phase factor is harmless); then, set $h_j(z) = (\psi_z^j \,|\, \mathrm{Op}(\mathfrak{S})\,\psi_z^j)$. If

$$\mathcal{E} = (2i\pi)^{-1}\left(x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi} + 1\right),$$

one then has the pair of equations

$$\left(\Delta - \frac{1}{4}\right)\left(z \mapsto (\psi_z^j \,|\, \mathrm{Op}(\mathfrak{S})\,\psi_z^j)\right) = (\psi_z^j \,|\, \mathrm{Op}(\pi^2\,\mathcal{E}^2\,\mathfrak{S})\,\psi_z^j)\,, \tag{12}$$

a formula which reduces the study of the spectral decomposition of Δ (in particular in the automorphic situation) to that of the decomposition of distributions on $\mathbb{R}^2$ into their homogeneous components.

One can then develop a symbolic calculus of the operators with automorphic symbols, and the composition formulas make use of all the main tools of the theory of non–holomorphic modular forms: Maass eigenforms and Eisenstein series, L–series, Hecke operators, convolution L–functions, triple products... We can only refer the interested reader to the quoted book for this rather complicated quantization theory. Let us stress that, just like the automorphic Laplacian, the automorphic Euler operator $\mathcal{E}$ on $SL(2,\mathbb{Z})\backslash\mathbb{R}^2$ has both a continuous and a discrete spectrum: the composition formulas involve both convergent series of *cusp–distributions* and integral superpositions of *Eisenstein distributions*.

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